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Fuzzy Mathematics: From Philosophy to Abstract Theorem

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Abstract: ***A Poland philosopher, in 1920s, introduced the concept of three-valued logic which was later generalized to four-valued logic and finally to infinite-valued logic. This infinite-valued logic was the idea behind the concept of graded membership initiated by Zadeh (1965) which further gave birth to fuzzy logic. Nowadays, this concept of fuzziness is a popular field of research due to its application oriented significance.The aim of the present paper is to give a very brief overview of this development - from philosophical thought to theabstract theorems.***

Keywords: Fuzzy logic, two-valued logic, multi-valued logic, infinite -valued logic, fuzzy metric spaces, fuzzy normed linear spaces, fuzzy Hilbert spaces

1. Introduction

As per Oxford dictionary, the literal meaning of the word 'Fuzzy' is "frizzy, blurred, indistinct" whileMathematicsis defined as "the abstract science of number, quantity, and space, either as abstract concepts (pure mathematics), or as applied to other disciplines such as physics and engineering (applied mathematics)". Then,how this 'abstract science'could be fuzzy? The topic'Fuzziness in Mathematics' seems paradoxical! But actually it's not.The present paper elaborates it.

In the second section,attention of the reader is drawn to the relationship of mathematics and philosophy.The third section is about fuzzy logic. What is fuzzy logic and how it is to be tackled, is the point of discussion in this section. The fourth section gives a brief overview of the development of fuzzy spaces.

The details given here are just the outlines. Fuzzy algebraic theory is equally evolved but this paper is focussing mainly on fuzzy functional analysis. Lakhs of results are available in literature but space limitation compels the author to just touch the conceptsor at times just look away and move further. But, the author feels that thegiven information is sufficient to inspireacurious researcher.In the last section, some avenues for future research are suggested.

2. Philosophy of Mathematics

It is always believed that there is a close connection between Mathematics and Philosophy. According to one set of philosophers, mathematics is eternal and philosophy discovers it while the others feel philosophical thoughtsare initial thoughts and mathematics follows them. Which one is following the other, is a bone of contention till date.

According to Plato (c.428-347 B.C.), a philosopher, Mathematics is a direct description of reality. The objects of pure geometry i.e. points, lines, circles etc. are independently existing eternal objects and Plato called them as *forms.* As per Platonic beliefs, even before our birth, our souls have direct interaction with these *forms* though we forget most of them because of our traumatic experience of taking birth. While growing, we recapture this forgotten knowledge by the dialectical process. The Aristotelians (c.384-322 B.C.) consider the *forms*identified by Platonistsas abstractions of our experiences. Accordingly, two philosophical theorists have emerged-rationalists and empiricists. A lot more convincing arguments are available in the literature where the reliability of one theory is established over the other.

Immanuel Kant (1724-1804), another philosopher, tried to diffuse some of the differences of rationalists and empiricists. He introduced a different classification -*a priori* and *a posterior*. According to him, the mathematical propositions are *a priori* and the remaining truths are *a posterior. E.g.* the statement 'a circle is the locus of a point equidistant from the centre' is *a posterior* while the *form* 'circle' is *a priori*i.e.all mathematical truthsare *a priori*. Solar system was already there so in a way, it's*a priori*butthe astronomers discovered it and traced their path to be an elliptic orbit, makes it *a posterior*because this conclusion was drawn and accepted only after a lot of arguments, counter-arguments, discussions of physicists and mathematicians.So drawing a line between *a priori* and*a posterior*becomes increasingly difficult. The German mathematician and philosopherGottlobFrege (1848-1925)argued that the truths of these *a priori* propositions could only be established by valid logical arguments. For all these logical arguments, the definitions and the concepts of mathematics proved to be quite useful. He used concepts from mathematics for analysis of sentences. His thoughtful choice of notations and concepts of mathematics and linguistics, created a very powerful system of reducing more of mathematics to logic. According to Russell [53] "logic has become more mathematical and mathematics has become more logical… They differ as boy and man: logic is the youth of mathematics and mathematics is the manhood of logic". The Italian mathematician G. Peano (1858-1947), English philosopher Alfred North Whitehead (1861-1947) and Bertrand Russell (1872-1970) did brilliant work in the direction of symbolic-logic (also called deductive logic) which takes care of linguistic imprecision using set theory.

3. Fuzzy Logic

Quoting Copi and Cohen [14] “Logic is the study of the methods and principles used to distinguish correct reasoning from incorrect reasoning”. Thomson[56] believes “Logic is the science of the laws of thought, the process of thought, or that active function of the mind by which impressions received are described, classified, and compared”. This mental activity is always subject to space, time, mind sets and hence may not be precise and certain. Thus, the logical thought of one may seems fuzzy to others. This fuzziness needs to be measured or minimized which has become possible,nowadays, due to ‘Fuzzy Logic’.

In fuzzy logic, actually, there is nothing fuzzy, rather it is a precise logic of imprecision and approximate reasoning. It is a form of multi-valued logic derived from fuzzy set theory to deal with problem of approximation rather than precision. It is a system of logic for dealing vague concepts. Zadeh [59] views it as an attempt at formalization/mechanization of the following two human capabilities:

- The capability to converse, reason and make rational decisions in the environment of imprecision, uncertainty, incompleteness of information, conflicting information, partiality of truth and partiality of possibility.
- The capability to perform a wide variety of physical and mental tasks without any measurements and computations.

Let us understand this imprecision via differentexamples:
(i) According to mathematical laws,$\frac{1}{2} + \frac{1}{2} = 1$ but let us take ½ apple and ½ orange. Could anyone get here some well-defined object as the resultant? The resultant ‘one’ is neither an apple nor an orange. Even if we take two ½ portions of the same apple, we cannot get ‘one apple’ again because it has been cut into two pieces and for humans it is really impossible to join these two pieces to get the one of the original form i.e. uncut apple which means the above stated mathematical law is totally failed! The reason being, ½, in the mathematical law is just a fractional number while in the statement ‘½ apple and ½ orange’ it’s an adjective for the objects in reference and we are comparing a numbers with adjectives.

(ii) 8 + 5 = 13. But is it true always? In modulo arithmetic 8+ 5 can be any number depending on the choice of modulo number n. e.g.
8+5 = 1 if n=12 and
8+5= 7 if n=6.
i.e. the mathematical laws are not universally true. They need additional specific descriptions also.

(iii) The set of numbers 'close to 5' may contain 5.01, 5.05, 100, 1000, 5000…. as each one, in the descending order, is much closer to its predecessor i.e. 100 is much closer to 5 as compared to 1000 and so on. Here the word ‘close’ is ambiguous.

(iv) Consider another example, due to Lenmann and Cohn [44] of class of chairs. The object 'chair' has some specific characteristics like at least it has leg/s, a seat to sit and a back. So the class of chairs may include typical four-legged chairs, wheel chairs, rocket chairs, folding chairs etc. But are they all identical? No, rather we may grade them according to their characteristics/ shapes–starting from closest to the description given to the end which is total mismatch to the specified characteristics. As per the specifications, foot stool with four steps and a western toilet seat, both should also be members of the given class but would anyone is willing to consider these two as chairs?

(v) ‘If a car is moving at a high speed then the brakes are to be applied slowly’. Here the words 'high' and 'slow' are imprecise.On highways speed of 60 is not considered as high while forlocal roads it’s high.

In each of the above examples, the language description is imprecise or insufficient to describe the setsand classes or the law and concept itself, which at times put even so called ‘certain and perfect’ mathematicallawsat stake.In physical world,always there is a spark or thought in the mindwhich is then narrated or translated via language and hence a mathematical model came into existence. This process adds up imprecision in understanding and hence in conversion. The philosopherJohn F. Sowa endorses this by saying“no language with a finite vocabulary can have a one to one mapping of words to every aspect of every topic”.

There are many areas, mostly among social sciences where natural language is preferred being more understandable as compared to formal language of mathematics, and hence these areas are much prone to such imprecisions which in turn creates ambiguities in problem analysis and in case studies.Quoting Zadeh[58] “more often than not, the classes of objects encountered in the real physical world do not have precisely defined criteria of membership”. This gap between mental perception and reality generates vagueness, imprecision, uncertainty which is evident in thepreviously discussed examples.

Among modern age philosophers, the very first one was Peirce [50] who identified this vagueness. Russell [52] too endorsed it by saying “Everything is vague to a degree which you do not realize until they tried to specify”. To deal with this imprecision and uncertainty, challenge is to devise some method to get rid of it or to get it minimized and here fuzzy logicplays an important role.

Actually, fuzzy logic is the generalization of the classical two-valued logic. In two valued logic any assertion or given proposition is either true (T) or false (F) i.e. T and F are complement to each other. If falsity of a proposition is denoted by 0 and truth by 1 then $\overline{0} = 1$and then $\overline{1} = 0$ where upper bar represents negation and there is no space for 'possibility'. Fuzzy logic created space for this third region- the possibility.

It was Plato (428-347 B.C.) who indicated the third region ‘possibility’ beyond T and F initially. Though, in the eastern world like India and China, logic was always considered as multi-valued but in the western world a systematic alternative to this prevalent bi-valued logic of

Aristotle, was first proposed by a Polish Philosopher J. Łukasiewicz (Borkowski,[46] in his 'farewell lecture'. It was three valued logic, which was published in Polish language in 1920. He is known as the founder of multi-valued logic. Initially, this third region 'possible' was assigned the value 0.5 with$\overline{0.5} = 0.5$. Different logicians used varied rules for 'conjunction' and 'disjunction' and hence generated different patterns.Klir and Yuan[42]comparedfive of the then best known such patterns in his book and showed that all generate different truth tables. In 1937, Max Black [11] proposed so-called 'consistency profiles', which are considered as ancestors of fuzzy membership function and characterized vague symbols. Weyl [57] replaced traditional characteristic function by a continuous characteristic function. Similar generalizations were proposed by Kaplan and Schott [36] in 1951. The notion of infinite-valued logic is thecontribution ofZadeh. He proposed that the whole range of real numbers in [0,1] may be the truth set for any proposition. Considering the colour of some object to be 'white' compared to the colour 'black' can have many stages as the addition of the slightest of the black colour to the white will give rise to a new colour 'lesser white' and 'more black' and hence could be graded with infinite values.These graded membership functions have been identified as fuzzy sets.Zadeh [58], in his path-breaking research paper,defined many laws involving these fuzzy sets which are obvious extensions of laws of classical sets.

While creating a model for any practical purpose, the information comes mainly from two resources – one source is human experts who use natural language and the other is sensory measurements and mathematical workings. A scientist or engineer is supposed to combine these two to create a mathematical model i.e. human knowledge is to be transformed into a mathematical model incorporating all possible imprecisions of the natural language and machine errors. Here the fuzzy systems play a vital role because in fuzzy systems,mostly, IF-THEN rules are used. These IF-THEN statements could be characterized by membership functions e.g. in the example (v) above 'high' and 'slow' could be considered as two membership functions defined on the speed of the car and accordingly a mathematical model could be framed to analyse the situation.

But the concept of fuzziness remained scepticalfor long time and ridiculed by many academicians.Zadeh[59] mentioned a few of his criticisms due to the then statisticians, logicians, computer scientists, mathematicians etc. Despite this prevailing criticism, the idea of Fuzzy Sets fascinatedmany scholarsand inspired them to dedicate themselves to this field. Zadeh himself continued his research in this direction and wrote more than 50 research papers. In fact, a new area of research has been defined

Today,theuseof this 'one time sceptical' fuzzy theory is evident in the physical world inthe form of luxurious as well as essential gadgets like washing machines, self-driven cars, subway trains, microwaves, autofocus cameras, speech recognition, pattern recognition, fingerprint based technology like biometry, and so on. Whilethese applications are visible to common man in the physical world, the computer scientists and mathematicians are still busy in digging up further to create/get more and more out of this theory. But every application module has, at the back some pure mathematical concepts which mostlyremainhidden to the application developers and users. So it is impossible that the applied fuzzy theory develops day by daybut pure mathematics remains unaffected anduntouched!

One of these developments in the area of pure mathematics i.e. regarding fuzzy spacesis outlined here in the coming section.

4. Development of Fuzzy Spaces

The first very systematic book, collecting many ofscattered ideas regarding fuzzy concepts of that time, is by Kaufmann [41] (French version-1973 with English translation-1975). This book is in two volumes.The first volume is about the theoretical concepts while the second one deals with the applications. Approximately at the same time, around 1974, Neġoiţă and Ralescu [49] also published a monograph in fuzzy theory which was in Romanian language. Its English version, translated by the authors themselves,came in 1975.While Kaufmann discussed fuzzy graphs, fuzzy relations and laws of fuzzy compositions, Neġoiţăand Ralescu included some additional fuzzy structures like fuzzy categories, fuzzy topological spaces, fuzzy programming, fuzzy algorithms etc. They also introduced the concept of L-flou sets which have been shown to be equivalent to the concept of L-sets introduced by Goguen[26]. Chang [12] extended the concept of fuzzy sets to define fuzzy topological spaces. The concept of fuzzy groups is due to Rosenfeld [51]. Some other referral books are Dubiosand Prade[18], Zimmermann [60] and Klir and Yuan [42]. But, till date, most of the significant research work in this area is confined to research papers only.

In early days of development of Fuzzy Metric Spaces, mainly there were two prevailing directions. Some scholars took interest in dealing with topological properties of fuzzy metric spaces while the others focused on the distance between the objects. The torch bearers in the first direction are Kramosil and Michalek[43], Erceg [22] and [23], Deng [15 and [16],Artico and Moresco[3], [4], and [5], Hu [32] while significant contributors in the second direction are Kaleva and Seikkala[35], Kaleva[33] and [34],Ekland and Gähler[19] and [20]. A very comprehensive study ofthese developments regarding fuzzy topological spaces is by Shostak[55]. The book of Diamond and Kloedon[17] is also of worth mentioning.

Katsaras[37] was the first one who introduced the idea of fuzzy norm on a linear space systematically. Felbin[24] introduced fuzzy norm using fuzzy intervals. The fuzzy metric corresponding to Felbin fuzzy norm is the same as that ofKaleva and Seikkla[35]. Another fuzzy norm defined by Cheng and Moderson[13] corresponds to the fuzzy metric of Kramosil and Michalek[43].Inspired by this norm, Bag and Samanta[6] defined fuzzy normed linear space using a more generalisedfuzzy norm. They used 'min' condition of the t-norm and associated with it a family of crisp norms called α-norms. Subsequently, a comparison of all previously defined norms was established byBag and Samanta [7] in 2008. The definition of fuzzy anti-norm using 'max' condition of t-normsand its relationship with crisp norms is the significantly new concept in this paper. In 2013, Bag and Samanta [8]further improved their definition of fuzzy norm of 2003 by relaxing previously implemented 'min' condition of the t-norm. Independently, Gähler and Gähler[25] gave another definition of fuzzy norm and discussed proper fuzzy normed vector spaces and proper fuzzy normed algebras.

The earliest known definition of fuzzy vector space is due to Katsaras and Liu [38] and Lubczonok[45],Biswas [10], El-Abyad and El-Hamouly[21] and Kohli and Kumar [39] and [40], Mazumdar and Samanta[47] defined fuzzy inner product spaces in their own ways. In 2009, Goudarzi and Vaezpour[29] introduced fuzzy Hilbert spacesas a modified version of their own definition given in 2008. Hasankhani, Nazari, Saheli[30] defined the fuzzy Hilbert space following fuzzy norm of Felbin[24].Another approach is due to Mirzavaziri and Moslehian[48]. Corresponding to the improved version of fuzzy norm, due to Bag and Samanta[8],one may givea newversion of definition of Hilbert space very soon.

5. Concluding Remarks

Using these fuzzy spaces, thousands of results of classical topology and classical functional analysis have been proved and lakhs are yet to be looked at to create their fuzzy analogues. Many scholars have chosen spaces as ordinary spaces but mappingsdefined on them as fuzzy mappings. Heilpern[31] was the first who proved Banach Contraction Principle for a fuzzy contraction mapping using special fuzzy sets called approximate quantities. The author herself has published a few such fuzzy analogues in her research papers Arora and Sharma [1] and [2], Sharma [54].One may consider fuzzy spaces as well as fuzzy mappings defined on these fuzzy spaces to explore fuzzy analogues of traditional functional analysis. Very recently the notion of operator's fuzzy norm has also been given by Bag and Samanta[9] and surely, very shortly, we may expect several fuzzy analogues of classical operator theory also. The journey is still on…

Since the ‘fuzzy theory’ is in the developing stage so the definitions and concepts are ought to change every now and then, giving rise tomuch scope of research, may be in the field of fuzzyfunctional analysis or fuzzy operator theory or fuzzy programming or any other branch. Keen and interested researchers have a vast field to play- in applied mathematics as well as in pure mathematics.

6. Summarizing

There was a philosophical thought- theLogic,or we can say thesymbolic-logic, which adapted Set Theoretic concepts of mathematics and from there Zadeh, the founder of Fuzzy Theory, ignited thousands of brains to create a lot more literature in this hugely beneficial and successful Fuzzy Theory – i.e. we have moved from aphilosophical THOUGHT to abstract THEOREMSsailing through the turbulent stream of so called FUZZINESS!!

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